Runge-Kutta Time Integration and High-Order Spatial Discretization – a New Dynamical Core for the LMK

Jochen Förstner, Günther Doms
Deutscher Wetterdienst, Frankfurter Straße 135, 63067 Offenbach am Main
E-mail: jochen.foerstner@dwd.de, guenther.doms@dwd.de

LMK is the name for a new development branch of the *LM* aiming at the meso-gamma scale (horizontal resolution of 2-3 km) and shortest range ("*K*ürzestfrist") forecasts periods (3-18 h). The new dynamical core for the LMK is based on different variants of 2-timelevel Runge-Kutta schemes which are combined with a forward-backward scheme for integrating high-frequency modes of the elastic equations. The first one is the 3rd-order Runge-Kutta scheme used by Wicker and Skamarock (2002) whereas the second one is a total variation diminishing (TVD) variant of 3rd-order (Liu, Osher, and Chan 1994).
For horizontal advection upwind or centered-differences schemes of 3rd- to 6th-order can be used – the operators are formulated in advection form. The vertical advection is normally treated in an implicit way using a Crank-Nicolson scheme and centered-differences in space.
Most slow tendencies such as vertical diffusion, thermal/solar heating, parameterized convection, coriolis force and buoyancy are computed only once using values of the prognostic variables at time step n. These tendencies are fixed during the individual Runge-Kutta steps and contribute to the total slow-mode tendencies which are integrated in several small time steps together with the fast-mode tendencies in a time-splitting sense. In contradiction to this, the whole 3D-advection is computed in each of the Runge-Kutta steps.
In the following the procedure for the TVD-Runge-Kutta scheme is described mathematically in a simplified form – the treatment of the physical forcing terms is omitted and the only operators listed are the ones for advection.

$$\textit{Problem to Solve:} \qquad \frac{\partial \phi}{\partial t} = L^{slow}(\phi) + L^{fast}(\phi)$$

TVD-variant of 3rd-order Runge-Kutta:

$$\phi^{*}_{i,k} = \phi^{n}_{i,k} - \Delta t\ L^h_i(\phi^n) - \Delta t\Big(\beta^+ L^v_k(\phi^*) + \beta^- L^v_k(\phi^n)\Big) \qquad = \phi^0_{i,k} + \Delta t\ L^{slow}_{i,k}\Big|_0^{*}$$

$$\phi^{**}_{i,k} = \frac{3}{4}\phi^{n}_{i,k} + \frac{1}{4}\phi^{*}_{i,k} - \frac{1}{4}\Delta t\ L^h_i(\phi^*) - \frac{1}{4}\Delta t\Big(\beta^+ L^v_k(\phi^{**}) + \beta^- L^v_k(\phi^*)\Big) \qquad = \phi^0_{i,k} + \frac{1}{4}\Delta t\ L^{slow}_{i,k}\Big|_0^{**}$$

$$\phi^{n+1}_{i,k} = \frac{1}{3}\phi^{n}_{i,k} + \frac{2}{3}\phi^{**}_{i,k} - \frac{2}{3}\Delta t\ L^h_i(\phi^{**}) - \frac{2}{3}\Delta t\Big(\beta^+ L^v_k(\phi^{n+1}) + \beta^- L^v_k(\phi^{**})\Big) \qquad = \phi^0_{i,k} + \frac{2}{3}\Delta t\ L^{slow}_{i,k}\Big|_0^{n+1}$$

Time-Splitting Method:

1. step:

$$\phi^{0+\Delta\tau}_{i,k} = \phi^0_{i,k} + \Delta\tau\ L^{fast}_{i,k}(\phi^0) + \Delta\tau\ L^{slow}_{i,k}\Big|_0^{\times}$$

remaining steps:

$$\phi^{\tau+\Delta\tau}_{i,k} = \phi^\tau_{i,k} + \Delta\tau\ L^{fast}_{i,k}(\phi^\tau) + \Delta\tau\ L^{slow}_{i,k}\Big|_0^{\times}$$

with $\times = *,\ **$ and $n+1$ in the individual Runge-Kutta steps.

Horizontal and Vertical Operators:

$$L^h_i(\phi)^{\,(4\text{th})} = \frac{u_i}{12\Delta x}\Big[\phi_{i-2} - 8\big(\phi_{i-1} - \phi_{i+1}\big) - \phi_{i+2}\Big]$$

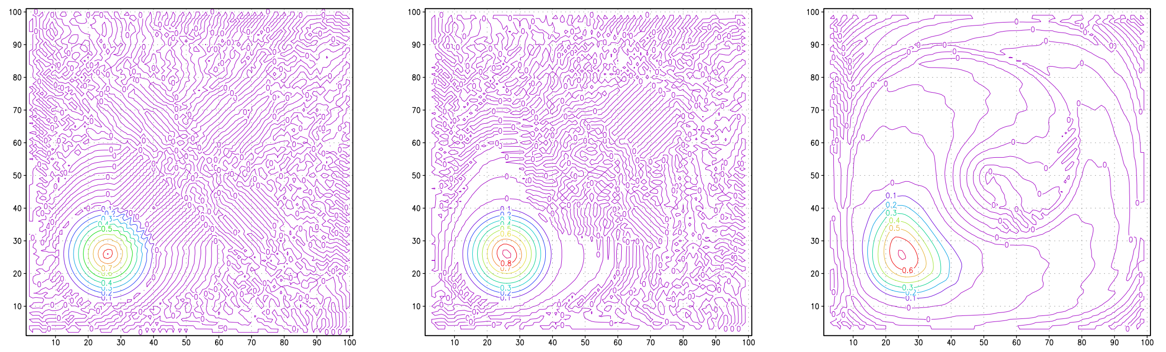
$$L^h_i(\phi)^{\,(6\text{th})} = \frac{u_i}{60\Delta x}\Big[-\phi_{i-3} + 9\big(\phi_{i-2} - \phi_{i+2}\big) - 45\big(\phi_{i-1} - \phi_{i+1}\big) + \phi_{i+3}\Big]$$

$$L^h_i(\phi)^{\,(5\text{th})} = L^h_i(\phi)^{\,(6\text{th})} + \frac{|u_i|}{60\Delta x}\Big[-\phi_{i-3} + 6\big(\phi_{i-2} + \phi_{i+2}\big) - 15\big(\phi_{i-1} + \phi_{i+1}\big) + 20\phi_i - \phi_{i+3}\Big]$$

$$L^v_k(\phi)^{\,(2\text{nd})} = \frac{w_k}{2\Delta z}\big(\phi_{k+1} - \phi_{k-1}\big) + M^v_k(\phi) \qquad \text{with } M^v_k(\phi)\text{: vertical turbulent mixing term.}$$

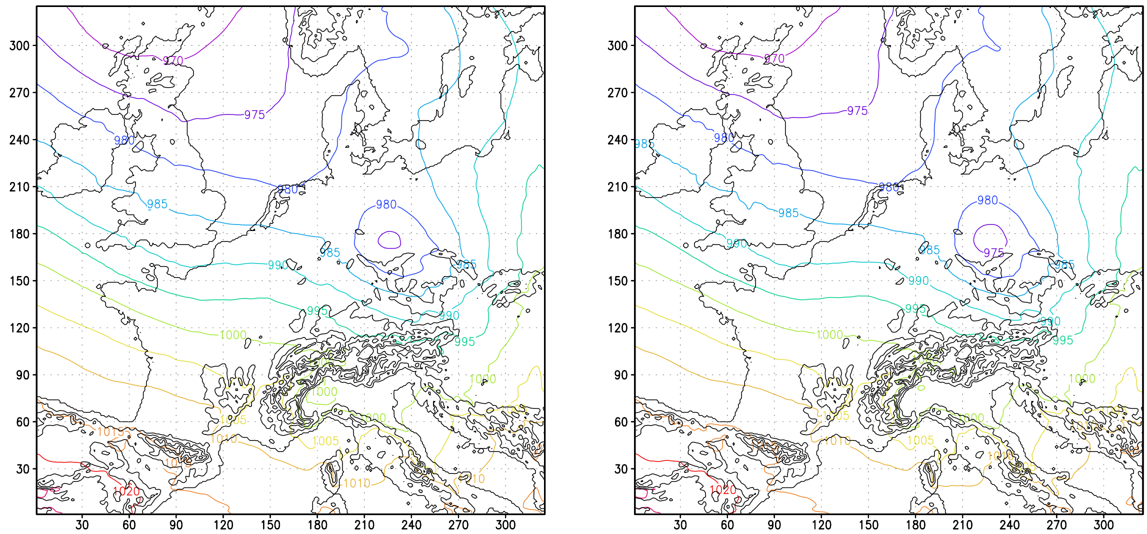
Results of an advection test problem of a tracer in a deformational flow field (Durran 1999) are given in Figure 1. The number of time steps used for the stable integration of one deformation cycle is given in the caption for each of the different schemes. The initialized field was a cone with a maximum of 1.0 and a radius of 15 grid spacings.

To test the robustness of the scheme, the winter storm case "Lothar" (26 December 1999) was simulated with the LM. The maximum horizontal velocity during the simulation reaches 108 m/s. For this case the new scheme in the combination TVD-RK-3rd/UP-5th allows a time step of 72 s at a resolution of 7 km compared to a time step of 40 s of the operational Leapfrog/CD-2nd scheme. Results are shown in Figure 2.



(a) RK-3rd / CD-4th – 670 time steps. (b) TVD-RK-3rd / CD-4th – 450 time steps. (c) TVD-RK-3rd / UP-5th – 380 time steps.

Figure 1: *Advection of a tracer in a nondivergent deformational flow (Durran 1999). Simulation results after one deformation cycle.*



(a) Leapfrog / CD-2nd – $\Delta t = 40$ s. (b) TVD-RK-3rd / UP-5th – $\Delta t = 72$ s.

Figure 2: *Winter storm "Lothar": 28 hour forecast of the mean sea level pressure in hPa for 26 December 1999, 16 UTC.*

References

Durran, D. R. (1999). *Numerical Methods for Wave Equations in Geophysical Fluid Dynamics*. New York: Springer.

Liu, X.-D., S. Osher, and T. Chan (1994). Weighted essentially non-oscillatory schemes. *J. Comput. Phys. 115*, 200–212.

Wicker, L. J. and W. C. Skamarock (2002). Time-splitting methods for elastic models using forward time schemes. *Mon. Wea. Rev. 130*, 2088–2097.