SPECTRAL ANALYSIS OF REAL-VALUED FUNCTIONS ON A SPHERE: VISUALIZATION OF SPATIAL SPECTRA

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An expansion of real functions in double Fourier series using associated Chebyshev polynomials was proposed in [1]. This expansion, in contrast to the traditional one based on spherical harmonics, is uniformly convergent on the globe, including the poles, and allows the use of Fast Fourier Transform in both directions [1, 2].

The associated Chebyshev polynomials of the first kind are simply depending on latitude. They can be expressed as Fourier series in terms of cosine polynomials for even zonal wave numbers *m* and as Fourier series in terms of sine polynomials for odd zonal wave numbers *m* [1]. Thus, the spatial spectrum of any real–valued function has a clear physical meaning; in particular, it can easily be visualized on a plane.

Let a real-valued function $f(\theta, \lambda)$ be defined on the surface of a sphere Ω $(0 \le \theta \le \pi,\ 0 \le \lambda \le 2\pi)$ and its square be integrable with some weight, i.e., $f(\theta, \lambda) \in L_2(\Omega)$. The function $f(\theta,\lambda) \in L_2(\Omega)$ is represented in the form of a double Fourier series:

$$f(\lambda,\theta)=\sum_n\left[d_n(\theta)\cos(n\lambda)+b_n(\theta)\sin(n\lambda)\right],\ b_0(\theta)\equiv 0. \qquad (1)$$

The Fourier coefficients $d_n(\theta)$ and $b_n(\theta)$ are continuous functions of θ and can be uniformly fit by associated Chebyshev polynomials ($m=0,1$). Furthermore, we make the following transformation of (1):

$$f(\lambda,\theta)=\sum_n\left[d_n(\theta)\cos(n\lambda)+b_n\sin(n\lambda)=\sum_n C_n(\theta)\cos(n\lambda-\eta_n(\theta)\right],$$
$$C_n(\theta)=\sqrt{d_n^2(\theta)+b_n^2(\theta)},\quad \eta_n(\theta)=sign(b_n(\theta))ar\cos\left(d_n(\theta)\big/C_n(\theta)\right), \qquad (2)$$

where the amplitude $C_n(\theta)$ and the phase $\eta_n(\theta)$ are also continuous functions of θ and therefore can be expanded in Fourier series based on associated Chebyshev polynomials ($m=0,1$).

The appropriate Fourier series are

$$C_n(\theta)=\sum_m B_{n,m}\cos(m\theta),\quad \eta_n(\theta)=\sum_m\left(W'_{n,m}\cos(m\theta)+W''_{n,m}\sin(m\theta)\right),$$
$$C_n(\theta)=\sum_m A_{n,m}\cos(m\theta+\delta_m),\quad A_{n,m}=\left|B_{n,m}\right|,\ \ \delta_m=\frac{1-sign(A_{n,m})}{2}\pi. \qquad (3)$$

Then instead of (1) we have

$$f(\lambda,\theta)=\sum_m\sum_n A_{n,m}\cos\left(n\lambda+\sum_m\left(W'_{n,m}\cos(m\theta)+W''_{n,m}\sin(m\theta)\right)\right). \qquad (4)$$

As an example, we consider the spectrum of the 500 hPa geopotential height $H_{500}(\theta,\lambda)$. Eliminating the «climatic» component of $H_{500}(\theta,\lambda)$,

$$H'_{500}(\theta,\lambda)=H_{500}(\theta,\lambda)-\frac{1}{2\pi}\int_0^{2\pi}H_{500}(\theta,\lambda)\,d\lambda\ , \qquad (5)$$

we can calculate the spatial spectrum of $H'_{500}(\theta,\lambda)$) by using (3), (4). The values of $\hat{A}_{n,m}$ are shown in Fig. 1. Two power maximums corresponding to global $(m = n = 1-3)$ and synoptic $(m = n = 4-8)$ scales of the atmospheric circulation can be seen in the figure. The existence of these spikes is well known in meteorology, however, for the first time they were presented in such an obvious and pictorial form.

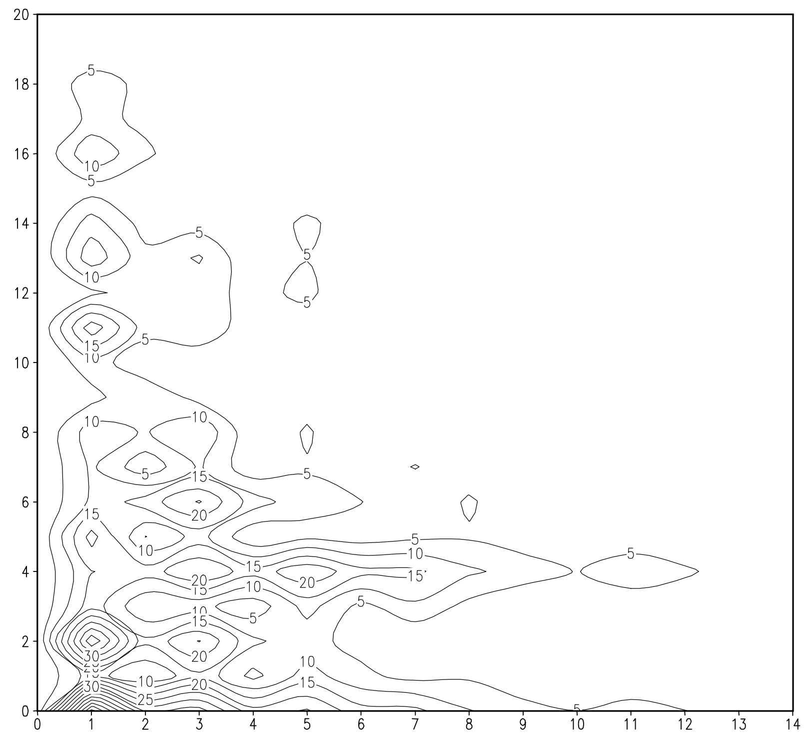

Figure 1. Distribution of amplitudes $\hat{A}_{n,m}$ of 500 hPa geopotential height deviations $H'_{500}(\theta,\lambda)$ at $m = 0,1,...,20$ (abscissa) and at $n = 0,1,...,20$ (ordinate).

Following the same method, the spatial spectrum of any real-valued function can be constructed in any latitude belt or even in a spherical trapezoid.

REFERENCES

1. ***Frolov, A.V. and Tsvetkov, V.I., On harmonic analyses of real-valued functions on a sphere, – Submitted to the Journal of Numerical Mathematics and Mathematical Physics, December 2003.***

2. ***Tsvetkov, V.I., Fast algorithm for harmonic analysis of real-valued functions on a sphere (this report).***